43rd Balkan Mathematical Olympiad
Thessaloniki, May 3 – 8, 2026
Official Problems and Solutions

Problem 1.

Find all integers $n \geq 4$ for which there exists a set S of n positive real numbers such that, for any $x, y, z \in S$ satisfying $x < y < z$, we have $\frac{z-x}{y} \in S$.

Proposed by Romania

Solution 1

We show that $n = 4$ is the only possibility. For $n = 4$, the set

$$S = \left\{ \frac{5}{9}, \frac{2}{3}, 1, \frac{11}{9} \right\}$$

works, since

$$\frac{1 - \frac{5}{9}}{\frac{2}{3}} = \frac{2}{3}, \qquad \frac{\frac{11}{9} - \frac{5}{9}}{\frac{2}{3}} = 1,$$

and

$$\frac{\frac{11}{9} - \frac{5}{9}}{1} = \frac{2}{3}, \qquad \frac{\frac{11}{9} - \frac{2}{3}}{1} = \frac{5}{9}.$$

From now on, assume $n \geq 5$.

Let

$$S = \{x_1 < x_2 < \cdots < x_n\}.$$

From the hypothesis, the following numbers belong to S:

$$\frac{x_n - x_{n-2}}{x_{n-1}} < \frac{x_n - x_{n-3}}{x_{n-1}} < \cdots < \frac{x_n - x_1}{x_{n-1}} < \cdots < \frac{x_n - x_1}{x_3} < \frac{x_n - x_1}{x_2}.$$

There are $2n - 5$ numbers here, hence

$$2n - 5 \leq n,$$

which gives $n \leq 5$. Therefore, no such sets exist with at least 6 elements.

It remains to exclude the case $n = 5$. In this case, we have $2n - 5 = n$, so the numbers above are exactly the elements of S. Hence

$$x_1 = \frac{x_5 - x_3}{x_4}, \qquad x_2 = \frac{x_5 - x_2}{x_4}, \qquad x_3 = \frac{x_5 - x_1}{x_4}, \qquad x_4 = \frac{x_5 - x_1}{x_3}, \qquad x_5 = \frac{x_5 - x_1}{x_2}.$$

From the first and the fourth equations, we obtain

$$x_5 = x_1 x_4 + x_3 = x_3 x_4 + x_1.$$

Thus

$$(x_3 - x_1)(x_4 - 1) = 0.$$

Since $x_1 \neq x_3$, it follows that

$$x_4 = 1.$$

Let

$$t := x_2 > 0.$$

Using the equations above, we get

$$x_1 = 2t(1-t), \qquad x_3 = 2t^2, \qquad x_5 = 2t.$$

Since $x_3 < x_4$, we get $2t^2 < 1$, and hence

$$t < \frac{1}{\sqrt{2}}.$$

Now note that

$$\frac{x_3 - x_1}{x_2} < \frac{x_4 - x_1}{x_2} < \frac{x_5 - x_1}{x_2} = x_5.$$

All three numbers belong to S, and the last one is x_5. Therefore, the first one cannot be x_4 or x_5, so

$$\frac{x_3 - x_1}{x_2} \in \{x_1, x_2, x_3\}.$$

Since

$$\frac{x_3 - x_1}{x_2} = \frac{2t^2 - 2t(1-t)}{t} = 4t - 2,$$

we obtain

$$4t - 2 \in \{2t(1-t), t, 2t^2\}.$$

The three cases give respectively

$$t = \frac{\sqrt{5}-1}{2}, \qquad t = \frac{2}{3}, \qquad t = 1.$$

But $t < \frac{1}{\sqrt{2}}$, so only

$$t = \frac{2}{3} \quad \text{or} \quad t = \frac{\sqrt{5}-1}{2}$$

can occur.

If $t = \frac{2}{3}$, then

$$S = \left\{ \frac{4}{9}, \frac{2}{3}, \frac{8}{9}, 1, \frac{4}{3} \right\}.$$

However,

$$\frac{x_4 - x_2}{x_3} = \frac{1 - \frac{2}{3}}{\frac{8}{9}} = \frac{3}{8} \notin S,$$

a contradiction.

If $t = \frac{\sqrt{5}-1}{2}$, then $t^2 = 1 - t$. Thus

$$\frac{x_4 - x_2}{x_3} = \frac{1-t}{2t^2} = \frac{1}{2}.$$

Also,

$$x_1 = 2t(1-t) = 2t^3 < \frac{1}{2} < t = x_2,$$

so $\frac{1}{2} \notin S$, again a contradiction.

Therefore, $n = 5$ is impossible. Hence, the only possible value is

$$n = 4.$$

Remark. In fact, all sets of 4 elements with the given property can be characterized. These are

$$\{a, b, 1, a+b\},$$

with $0 < a < b < 1$ and

$$a = 1 - b^2 \quad \text{or} \quad a = \frac{1}{b+1}.$$

For example, the choice $b = \frac{2}{3}$ and $a = 1 - b^2 = \frac{5}{9}$ gives

$$S = \left\{ \frac{5}{9}, \frac{2}{3}, 1, \frac{11}{9} \right\},$$

which is the example used above.

Problem 2.

Let n be a positive integer. A $2n \times 2n$ board is tiled with 2×1 and 1×2 dominoes. To *pivot* a domino is to select one of its two unit squares and rotate the entire domino by 90° clockwise, 90° anticlockwise, or 180° about the centre of that unit square. Prove that it is always possible to simultaneously pivot every domino such that, after all pivots have been performed, the dominoes still tile the board. *Proposed by the U.K.*

Solution 1

Claim: For any given domino tiling of the $2n \times 2n$ board, there exists another domino tiling in which no domino is common to both tilings.

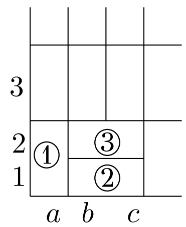
Proof of the claim. Divide the board into 2×2 squares. We can tile each of these 2×2 squares either with two vertical or two horizontal dominoes. The original tiling cannot have both a vertical and horizontal domino fully contained in the 2×2 square (as they would overlap). Hence, at least one choice (for each 2×2 square) will lead to a tiling with no domino in common with the original tiling. □

Now consider a bipartite graph G with vertex sets $X = \{x_1, \ldots, x_{2n^2}\}$ and $Y = \{y_1, \ldots, y_{2n^2}\}$ where x_i represent the dominoes in the original tiling and y_j represent the dominoes from the tiling in the Claim. Draw an edge between x_i and y_j if they have a square in common (by the Claim, they will either have 0 or 1 squares in common).

Each vertex will have degree 2 so we can decompose G into disjoint cycles, and since G is bipartite these cycles have even length. These cycles imply a perfect matching on G with say $x_i \in X$ paired with $y_{\sigma(i)}$[(1)].

The dominoes x_i and $y_{\sigma(i)}$ will share exactly one square so it is possible[(2)] to pivot x_i to $y_{\sigma(i)}$. Doing this for all dominoes in the original tiling gives a way of simultaneously pivoting them to get a new tiling of the board.

Remark: If we do not allow 180° domino rotations, then it is no longer always possible to pivot dominoes to make another tiling (aside from the trivial $n = 1$ case).



Indeed, in the figure, to cover a_1, we are forced to rotate domino 1 to squares a_1, b_1. Then, we are forced to rotate domino 2 to squares c_1, c_2. Then, we are forced to rotate domino 3 to squares b_2, b_3. But then there is no way to rotate the dominoes by 90° to cover square a_2.

(1)Comment: On each cycle $z_1 z_2 \ldots z_n$ match z_1, z_2 and z_3, z_4 and $z_5.z_6$... Since G is bipartite, the vertices $z_1, z_3, z_5 \ldots$ lie in X, whereas $z_2, z_4, z_6, \ldots$ lie in Y.

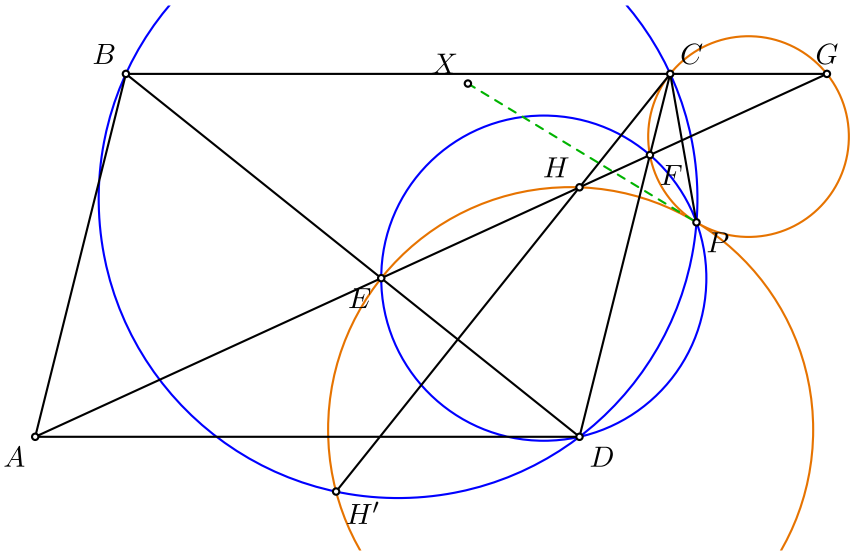
(2)Recall that $x_i, y_{\sigma(i)}$ are not the same domino, so such a suitable pivoting exists.

Problem 3.

Let $ABCD$ be a parallelogram with $\angle DAB < 90^\circ$ and $AB < AD$. Let H be the orthocentre of $\triangle BCD$ and H' be the reflection of H over line BD. Line AH intersects lines BD, CD, BC at E, F, G respectively. Prove the circumcircles of $\triangle HEH'$ and $\triangle CFG$ are tangent.

Proposed by the U.K.

Solution 1



Convention. Throughout this solution we use *ordinary, non-directed* angles, with the order of points and rays as in the figure.

Setup. Let P be the Miquel point of the lines BC, CD, DB, AH. By definition,

$$P \in (BCD), \qquad P \in (DEF), \qquad P \in (CFG). \tag{1}$$

We also use two standard facts: $H' \in (BCD)$ (the reflection of the orthocentre of $\triangle BCD$ across the side BD lies on the circumcircle), and C, H, H' are collinear (all three lie on the altitude from C, which is perpendicular to BD).

We will produce a line through P that is simultaneously tangent to (HEH') and to (CFG) at P; this proves the two circles are tangent at P.

Claim 1. $P \in (HEH')$.

Proof. Using (1), the fact that $H' \in (BCD)$, and the collinearity of C, H, H':

$$\begin{aligned}
\angle H'PE &= \angle EPD - \angle H'PD \\
&= \angle EFD - \angle H'CD && (P, E, D, F \text{ concyclic and } P, H', C, D \text{ concyclic}) \\
&= \angle CHF && (\text{angles of } \triangle CHF, \text{ with } E, H \text{ on line } AH) \\
&= \angle H'HE && (C, H, H' \text{ collinear}).
\end{aligned}$$

Hence P, E, H, H' are concyclic. □

Claim 2. *A, B, H, D are concyclic.*

Proof. Since H is the orthocentre of $\triangle BCD$, $BH \perp CD$ and $DH \perp BC$. Combined with $AB \parallel CD$ and $AD \parallel BC$, this gives $\angle ABH = \angle ADH = 90^\circ$, so A, B, H, D lie on the circle of diameter AH. □

Claim 3. *$\angle HEB = \angle CDH'$.*

Proof. Note that $\angle BHD = 180° - \angle BCD = 180° - \angle DAB$ (angles of the parallelogram $ABCD$ together with the fact that H is the orthocentre of $\triangle BCD$); this re-derives Claim 2 and shows that A, B, H, D form a cyclic quadrilateral.

Now we compute, using that B, E, D are collinear and A, E, H are collinear:

$$\begin{aligned}
\angle HEB &= 180° - \angle DBH - \angle BHE \\
&= 180° - \angle DBH - \angle BHA && (A, E, H \text{ collinear}) \\
&= 180° - \angle H'BD - \angle BDA && (H, H' \text{ reflections over } BD;\ A, B, H, D \text{ concyclic}) \\
&= 180° - \angle H'BD - \angle DBC && (AD \parallel BC, \text{ so } \angle BDA = \angle DBC) \\
&= 180° - \angle H'BC \\
&= \angle CDH' && (B, H', D, C \text{ concyclic}).
\end{aligned}$$

□

Claim 4. *Let X be a point on the tangent line to $(H'EHP)$ at P lying on the opposite side of the line CH from P. Then $\angle XPC = \angle PGC$.*

Proof. By the tangent–chord criterion in $(H'EHP)$ at P, with X on the indicated side,

$$\angle H'PX = 180° - \angle PEH'.$$

Combining this with the cyclic quadrilateral $PCH'D$ on (BCD) (which gives $\angle H'PC = \angle CDH'$), we compute:

$$\begin{aligned}
\angle XPC &= \angle H'PC - \angle H'PX \\
&= \angle CDH' - \big(180° - \angle PEH'\big) \\
&= \angle CDH' - 180° + \big(360° - \angle PEB - \angle BEH'\big) && (\text{angles around } E \text{ sum to } 360°) \\
&= 180° - \angle PEB + \angle CDH' - \angle BEH' \\
&= 180° - \angle PEB + \angle CDH' - \angle HEB && (H, H' \text{ reflections over } BD, \text{ with } B, E \in BD) \\
&= 180° - \angle PEB && (\text{Claim 3}).
\end{aligned}$$

Now using the collinearities B, E, D and D, F, C together with P, E, D, F concyclic on (DEF) and P, F, C, G concyclic on (CFG):

$$180° - \angle PEB = \angle PED = \angle PFD = 180° - \angle PFC = \angle PGC.$$

Hence $\angle XPC = \angle PGC$. □

Conclusion. By Claim 4, the equality $\angle XPC = \angle PGC$ is precisely the tangent–chord criterion for (CFG) at P: it says that the line XP is tangent to (CFG) at P. But XP was defined to be tangent to (HEH') at P (Claim 1 ensures $P \in (HEH')$). Hence XP is a common tangent of (HEH') and (CFG) at the common point P, so

$$\boxed{\text{the circumcircles of } \triangle HEH' \text{ and } \triangle CFG \text{ are tangent at } P.}$$

□

Remark (alternative final step). The conclusion of Solution 1 can also be reached by reversing the direction of the angle chase: instead of starting from the tangent to (HEH') at P and showing it is tangent to (CFG), one may start from the tangent to (CFG) at P and show it is tangent to (HEH'). We sketch this alternative below; it uses the same hypotheses on P as Solution 1 (that P is the Miquel point of the lines BC, CD, DB, AH, so $P \in (BCD), (DEF), (CFG)$, and $P \in (HEH')$ by Claim 1 of Solution 1).

Let t be the tangent to $\omega = (CFG)$ at P. By the tangent–chord theorem for ω, the angle between t and PC is

$$\angle PGC.$$

Thus it is enough to prove

$$\angle CPH' = \angle PGC + \big(180^\circ - \angle PEH'\big), \qquad (*)$$

because then the remaining angle between t and PH' is $180^\circ - \angle PEH'$, which is exactly the tangent–chord criterion for $\Gamma = (HEH')$ at P.

We now prove $(*)$. Since P, C, D, H' are concyclic, $\angle CPH' = \angle CDH'$. By Claim 3, $\angle CDH' = \angle HEB$. Since H, H' are symmetric with respect to BD and B, E, D are collinear, $\angle HEB = \angle BEH'$. Hence

$$\angle CPH' = \angle BEH'. \qquad (1)$$

On the other hand, using B, E, D collinear, D, F, C collinear, and the cyclic quadrilaterals P, E, D, F on (DEF) and P, F, C, G on (CFG),

$$\angle PGC = 180^\circ - \angle PFC = \angle PFD = \angle PED = 180^\circ - \angle PEB.$$

Therefore

$$\angle PGC + \big(180^\circ - \angle PEH'\big) = 360^\circ - \big(\angle PEB + \angle PEH'\big).$$

In the configuration, the rays EB, EP, EH' are arranged so that $\angle PEB + \angle PEH' = 360^\circ - \angle BEH'$, whence

$$\angle PGC + \big(180^\circ - \angle PEH'\big) = \angle BEH'.$$

Together with (1), this proves $(*)$, so t is a common tangent of Γ and ω at their common point P. □

Solution 2

Notation. All angles are taken as directed angles mod 180° and denoted by $\measuredangle$. Products of collinear segments used in power-of-a-point arguments are understood as directed products. Set

$$\Gamma = (HEH'), \qquad \omega = (CFG), \qquad \Omega = (BCD),$$

and let O, O_2 denote the centres of Ω, ω respectively. Let P be the *Miquel point* of the four lines BC, CD, DB, AH. By definition of the Miquel point,

$$P \in (BCD) = \Omega, \quad P \in (CFG) = \omega, \quad P \in (BEG), \quad P \in (DEF). \qquad (2)$$

We will prove that P is the common tangency point of Γ and ω.

Claim 1. $AH \parallel OC$.

Proof. Since H is the orthocentre of $\triangle BCD$, $BH \perp CD$ and $DH \perp BC$. Since $AB \parallel CD$ and $AD \parallel BC$, this gives

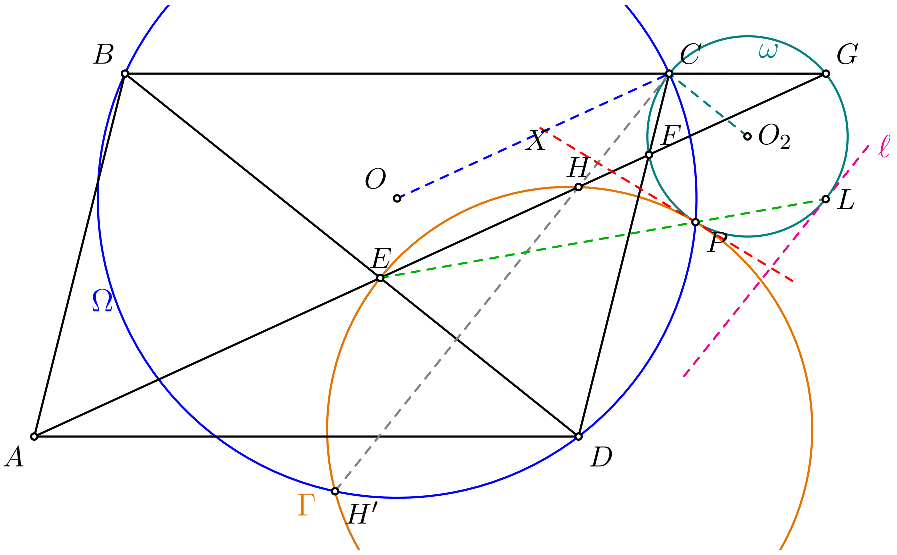
$$BH \perp AB, \qquad DH \perp AD,$$

so $\angle ABH = \angle ADH = 90^\circ$, and hence A, B, H, D are concyclic (on the circle of diameter AH). Therefore $\measuredangle HAB = \measuredangle HDB$. Since $DH \perp BC$, we have

$$\measuredangle HDB = 90^\circ - \measuredangle DBC.$$

By the tangent–chord angle applied to Ω at C, if τ is the tangent to Ω at C, then

$$\measuredangle(CD, \tau) = \measuredangle DBC.$$



Since $OC \perp \tau$,

$$\measuredangle OCD = 90^\circ - \measuredangle(CD, \tau) = 90^\circ - \measuredangle DBC = \measuredangle HAB.$$

Because $AB \parallel CD$, the equality $\measuredangle HAB = \measuredangle OCD$ forces $AH \parallel OC$. □

Claim 2. *The line CH is tangent to ω at C. In particular, $O_2C \parallel BD$.*

Proof. Since G, F, H, A are collinear and $CG \parallel AD$, we have

$$\measuredangle CGF = \measuredangle DAH.$$

Since A, B, H, D are concyclic, we have

$$\measuredangle DAH = \measuredangle DBH.$$

Since $BH \perp CD$, we get

$$\measuredangle DBH = 90^\circ - \measuredangle CDB.$$

On the other hand, since $CH \perp BD$ and $CF \parallel CD$, we have

$$\measuredangle HCF = 90^\circ - \measuredangle CDB.$$

Therefore

$$\measuredangle CGF = \measuredangle HCF.$$

By the tangent–chord theorem, CH is tangent to the circle (CFG) at C. □

Claim 3. *The Miquel point P lies on $\Gamma = (HEH')$.*

Proof. Recall two standard facts: $H' \in \Omega$ (the reflection of the orthocentre over a side of the triangle lies on the circumcircle), and C, H, H' are collinear (all lie on the altitude from C, which is perpendicular to BD).

Using (2) and the listed collinearities, we compute the following chain of directed-angle equalities:

$$\begin{aligned}
\measuredangle PEH &= \measuredangle PEF && (E, F, H \text{ collinear})\\
&= \measuredangle PDF && (P, E, D, F \text{ concyclic on } (DEF))\\
&= \measuredangle PDC && (D, F, C \text{ collinear})\\
&= \measuredangle PH'C && (P, C, D, H' \text{ concyclic on } \Omega)\\
&= \measuredangle PH'H && (C, H, H' \text{ collinear}).
\end{aligned}$$

Hence $\measuredangle PEH = \measuredangle PH'H$, so P, E, H, H' are concyclic, i.e., $P \in \Gamma$. □

Claim 4. $\measuredangle EPC = 90°$.

Proof. Since P, E, D, F are concyclic, we have

$$\measuredangle EPF = \measuredangle EDF.$$

Since P, F, G, C are concyclic, we have

$$\measuredangle FPC = \measuredangle FGC.$$

Moreover, by the previous claim,

$$\measuredangle FGC = \measuredangle FCH.$$

Therefore

$$\begin{aligned}\measuredangle EPC &= \measuredangle EPF + \measuredangle FPC \\ &= \measuredangle EDF + \measuredangle FGC \\ &= \measuredangle EDF + \measuredangle FCH \\ &= 90°.\end{aligned}$$

□

Claim 5. *Let L be the antipode of C on ω. Then E, P, L are collinear.*

Proof. Since CL is a diameter of ω and $P \in \omega$, $\measuredangle CPL = 90°$. By Claim 4, $\measuredangle EPC = 90°$. Thus the lines PE and PL are both perpendicular to PC at P; therefore they coincide, i.e., E, P, L are collinear. □

Claim 6. $EA^2 = EF \cdot EG$.

Proof. Since A, E, F, G are collinear (all on AH) and B, E, D are collinear:

(i) $\triangle ADE \sim \triangle GBE$. Indeed, $\measuredangle AED = \measuredangle GEB$, and since $AD \parallel BC$ with transversal AG, $\measuredangle DAE = \measuredangle BGE$. Hence

$$\frac{AE}{GE} = \frac{DE}{BE} \Longrightarrow AE \cdot BE = DE \cdot GE. \tag{i}$$

(ii) $\triangle ABE \sim \triangle FDE$. Indeed, $\measuredangle AEB = \measuredangle FED$, and since $AB \parallel CD$ with transversal AF, $\measuredangle BAE = \measuredangle DFE$. Hence

$$\frac{AE}{FE} = \frac{BE}{DE} \Longrightarrow AE \cdot DE = BE \cdot FE. \tag{ii}$$

Multiplying (i) and (ii) and cancelling the factor $BE \cdot DE$, $AE^2 = FE \cdot GE$, i.e., $EA^2 = EF \cdot EG$. □

Claim 7. *The inversion ι centred at E with radius EA preserves the circle ω, that is $\iota(\omega) = \omega$.*

Proof. The points F, G lie on ω and E, F, G are collinear, so the power of E with respect to ω equals $EF \cdot EG$, which by Claim 6 equals EA^2. This is exactly the square of the radius of ι. It is a standard fact that a circle not passing through the centre of an inversion is preserved by it precisely when the power of the centre with respect to the circle equals r^2. Hence $\iota(\omega) = \omega$. □

Claim 8. *The image $\ell := \iota(\Gamma)$ is a line perpendicular to BD passing through L; specifically, it is the tangent to ω at L.*

Proof. The circle $\Gamma = (HEH')$ passes through E (the centre of ι), so its image is a line ℓ. Since P, L, F, G all lie on ω, and since E, P, L and E, F, G are collinear, the power of E with respect to ω gives

$$EP \cdot EL = EF \cdot EG.$$

By Claim 6,

$$EP \cdot EL = EF \cdot EG = EA^2,$$

so $\iota(P) = L$. Since $P \in \Gamma$, this gives $L \in \ell$.

Because H, H' are symmetric across BD, the circle Γ is symmetric across BD, and since $E \in \Gamma \cap BD$, the centre of Γ lies on BD. Therefore the tangent to Γ at E is perpendicular to BD. It is a standard fact that the image of a circle through the centre of an inversion is a line *parallel* to the tangent to the circle at that centre. Hence $\ell \perp BD$.

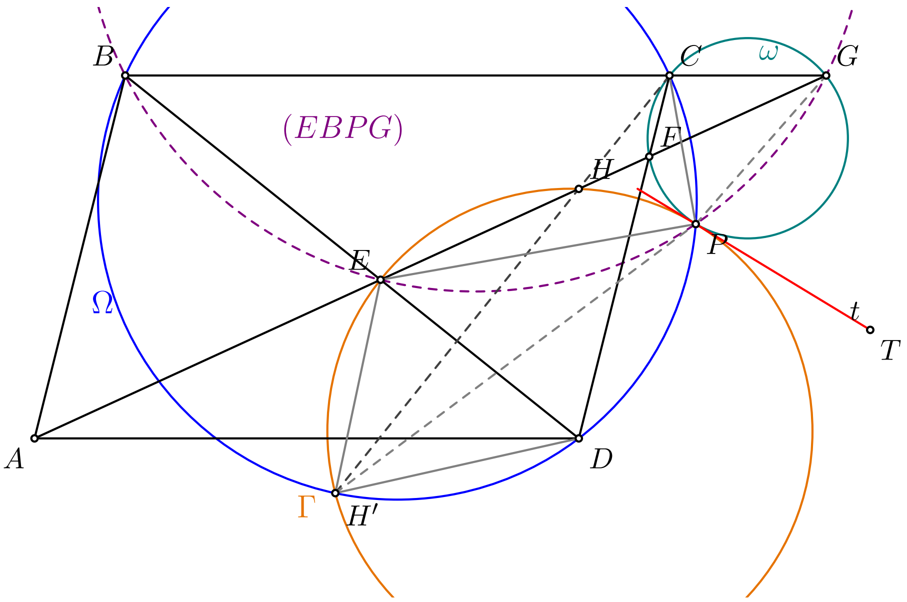
By Claim 2, $O_2C \parallel BD$, so the diameter CL is parallel to BD, and the tangent to ω at L is perpendicular to CL, i.e., perpendicular to BD. Thus ℓ and the tangent to ω at L are two lines through the same point L that are both perpendicular to BD. Therefore they coincide. □

Conclusion. By Claim 8, the line $\ell = \iota(\Gamma)$ is tangent to ω at L. Applying the inversion ι (which is involutive, $\iota^{-1} = \iota$) and using Claim 7 ($\iota(\omega) = \omega$), we deduce that $\iota(\ell) = \Gamma$ is tangent to $\iota(\omega) = \omega$ at $\iota(L) = P$. That is,

$$\boxed{\Gamma \text{ and } \omega \text{ are tangent at the Miquel point } P.}$$

□

Solution 3



We use directed angles modulo 180°, denoted by $\measuredangle$. Products of collinear segments, when used implicitly in cyclic or power arguments, are understood as directed products. We write

$$\Gamma = (EHH'), \qquad \omega = (CFG), \qquad \Omega = (BCD).$$

Recall the standard facts that $H' \in \Omega$ and that C, H, H' are collinear.

Define

$$P = (EHH') \cap (DH'BC), \qquad P \neq H'.$$

Since $H' \in (BCD)$, the circle $(DH'BC)$ is just $\Omega = (BCD)$. Thus

$$P \in \Gamma \qquad \text{and} \qquad P \in \Omega.$$

We shall prove that $P \in \omega$ and that Γ and ω are tangent at P.

Lemma. We have

$$\measuredangle H'EB = \measuredangle H'DC.$$

Proof of the lemma. Since H is the orthocentre of $\triangle BCD$,

$$BH \perp CD, \qquad DH \perp BC.$$

Together with $AB \parallel CD$ and $AD \parallel BC$, this gives

$$\angle ABH = \angle ADH = 90^\circ.$$

Hence A, B, H, D are concyclic, and therefore

$$\measuredangle(AH, AB) = \measuredangle(DH, DB).$$

Using $AB \parallel DC$ and the fact that H' is the reflection of H in BD, we obtain

$$\begin{aligned}\measuredangle(AH, BD) &= \measuredangle(AH, AB) + \measuredangle(AB, BD)\\ &= \measuredangle(DH, DB) + \measuredangle(DC, DB)\\ &= \measuredangle(DB, DH') + \measuredangle(DC, DB)\\ &= \measuredangle(DC, DH').\end{aligned}$$

This is

$$\measuredangle HEB = \measuredangle CDH'.$$

Reflecting in the line BD and using E, B, D collinear gives

$$\measuredangle H'EB = -\measuredangle HEB, \qquad \measuredangle H'DC = -\measuredangle CDH'.$$

Together with $\measuredangle HEB = \measuredangle CDH'$, this proves the Lemma. □

Claim 1. *The points E, B, P, G are concyclic.*

Proof. Since P, E, H, H' are concyclic, we have

$$\measuredangle EPH' = \measuredangle EHH'.$$

Since P, H', B, C are concyclic, we also have

$$\measuredangle H'PB = \measuredangle H'CB.$$

Therefore

$$\begin{aligned}\measuredangle EPB &= \measuredangle EPH' + \measuredangle H'PB\\ &= \measuredangle EHH' + \measuredangle H'CB\\ &= \measuredangle(EH, HH') + \measuredangle(H'C, CB)\\ &= \measuredangle(EH, CB).\end{aligned}$$

But E, H, G are collinear and B, C, G are collinear, so

$$\measuredangle(EH, CB) = \measuredangle EGB.$$

Thus $\measuredangle EPB = \measuredangle EGB$, and hence E, B, P, G are concyclic. □

Claim 2. *The points P, F, C, G are concyclic. In particular, $P \in \omega$.*

Proof. From P, B, C, D concyclic, we have $\measuredangle CPB = \measuredangle CDB$. From Claim 1, namely E, B, P, G concyclic, we have $\measuredangle BPG = \measuredangle BEG$. Therefore

$$\begin{aligned}\measuredangle CPG &= \measuredangle CPB + \measuredangle BPG\\ &= \measuredangle CDB + \measuredangle BEG\\ &= \measuredangle(DC, DB) + \measuredangle(DB, AH)\\ &= \measuredangle(DC, AH)\\ &= \measuredangle CFG,\end{aligned}$$

because D, B, E are collinear, while C, D, F and E, F, G are collinear. Therefore P, F, C, G are concyclic, so $P \in \omega = (CFG)$. □

Claim 3. *The circles Γ and ω are tangent at P.*

Proof. Let t be the tangent to $\omega = (PFCG)$ at P, and let T be any point on t with $T \neq P$. We prove that t is also tangent to $\Gamma = (PEHH')$ at P.

By the tangent–chord theorem applied to ω at P, for the chord PC, we have

$$\measuredangle CPT = \measuredangle CGP. \tag{1}$$

Since P, H', D, C are concyclic,

$$\measuredangle H'PC = \measuredangle H'DC. \tag{2}$$

Moreover, B, C, G are collinear and E, B, P, G are concyclic, so

$$\measuredangle CGP = \measuredangle BGP = \measuredangle BEP. \tag{3}$$

Using (1), (2), (3), and the Lemma ($\measuredangle H'DC = \measuredangle H'EB$):

$$\begin{aligned}
\measuredangle H'PT &= \measuredangle H'PC + \measuredangle CPT \\
&= \measuredangle H'DC + \measuredangle CGP \\
&= \measuredangle H'EB + \measuredangle BEP \\
&= \measuredangle H'EP.
\end{aligned}$$

By the tangent–chord theorem applied to $\Gamma = (PEHH')$, this means that t is tangent to Γ at P.

But t was already tangent to ω at P. Therefore Γ and ω have a common tangent at their common point P, and hence they are tangent at P.

Thus

$$\boxed{\text{the circumcircles of } \triangle HEH' \text{ and } \triangle CFG \text{ are tangent.}}$$

□

Solution 4

We use the following notation. Let

$$\alpha = \angle BDC = \angle ABD, \qquad \beta = \angle DBC = \angle BDA.$$

Let

$$O = CH' \cap BD.$$

Recall that $H' \in (BCD)$, since H' is the reflection of the orthocentre H of $\triangle BCD$ across BD, and that C, H, O, H' are collinear with

$$CO \perp BD.$$

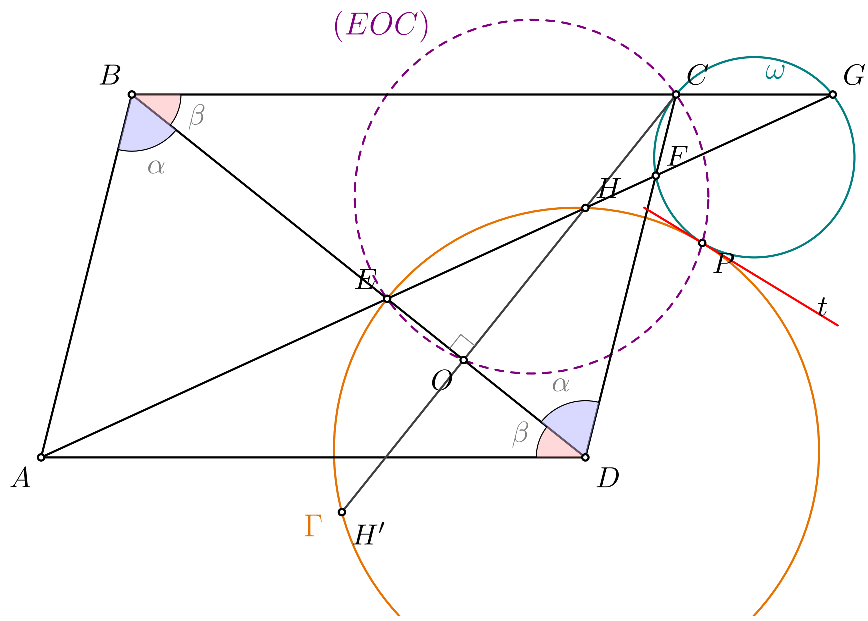
Let $P \neq E$ be the second intersection of the two circles

$$(EOC) \qquad \text{and} \qquad (EHH').$$

We shall prove that P is the tangency point of the circles (HEH') and (CFG).

Preliminary facts. Since H is the orthocentre of $\triangle BCD$, we have

$$BH \perp CD, \qquad DH \perp BC.$$



Together with

$$AB \parallel CD, \qquad AD \parallel BC,$$

this gives $\angle ABH = \angle ADH = 90^\circ$, so

$$A, B, H, D \text{ are concyclic.}$$

Therefore

$$\angle DAH = \angle DBH = 90^\circ - \alpha, \qquad \angle BAH = \angle BDH = 90^\circ - \beta. \tag{1}$$

Since $E \in BD$ and H, H' are symmetric with respect to BD, we have $EH = EH'$, and

$$\angle DEH = \angle BDA + \angle DAH = \beta + (90^\circ - \alpha).$$

Since $HH' \perp BD$, it follows that

$$\angle EHH' = 90^\circ - \angle DEH = \alpha - \beta.$$

Hence

$$\angle EHH' = \angle EH'H = \alpha - \beta. \tag{2}$$

Since $P \in (EOC)$ and $\angle EOC = 90^\circ$, we have

$$\angle EPC = 90^\circ. \tag{3}$$

Claim 1. *P, H', B, C are concyclic.*

Proof. Since P, E, H, H' are concyclic, by (2),

$$\angle H'PE = \angle H'HE = \alpha - \beta.$$

Using also (3), we obtain

$$\angle H'PC = \angle H'PE + \angle EPC = (\alpha - \beta) + 90^\circ. \tag{4}$$

Since H' is the reflection of H across BD, and $BH \perp CD$,

$$\angle H'BD = \angle HBD = 90^\circ - \alpha,$$

so
$$\angle H'BC = \angle H'BD + \angle DBC = (90^\circ - \alpha) + \beta. \tag{5}$$
From (4) and (5), $\angle H'PC + \angle H'BC = 180^\circ$, so P, H', B, C are concyclic. □

Claim 2. *P, E, B, G are concyclic.*

Proof. By Claim 1, $\angle BPH' = \angle BCH'$. Since $CH' \perp BD$, $\angle BCH' = 90^\circ - \beta$. By (2), $\angle EPH' = \angle EHH' = \alpha - \beta$. Hence
$$\angle BPE = \angle BPH' - \angle EPH' = (90^\circ - \beta) - (\alpha - \beta) = 90^\circ - \alpha.$$
By (1), $\angle BGE = \angle DAH = 90^\circ - \alpha$, so $\angle BPE = \angle BGE$ and P, E, B, G are concyclic. □

Claim 3. *P, C, F, G are concyclic.*

Proof. Since P, E, B, G are concyclic,
$$\angle EPG = 180^\circ - \angle EBG = 180^\circ - \beta.$$
Using (3),
$$\angle CPG = \angle EPG - \angle EPC = (180^\circ - \beta) - 90^\circ = 90^\circ - \beta. \tag{6}$$
Since C, F, D and F, G, A, H are collinear, $\angle CFG = \angle BAH = 90^\circ - \beta$ by (1). From (6), $\angle CPG = \angle CFG$, so P, C, F, G are concyclic. □

Claim 4. *The tangent to (CFG) at P is also tangent to (HEH') at P.*

Proof. Let t be the tangent to $(PFCG)$ at P. We prove
$$\angle HPF = \angle HEP + \angle FGP. \tag{7}$$
By (2), $\angle EPH = \angle EH'H = \alpha - \beta$. By Claim 3, $\angle CPF = \angle CGF = \angle BGE = 90^\circ - \alpha$. Using (3),
$$\angle HPF = 90^\circ - \angle EPH - \angle CPF = 90^\circ - (\alpha - \beta) - (90^\circ - \alpha) = \beta. \tag{8}$$
Since E, H, F, G are collinear, in triangle EPG,
$$\angle HEP + \angle FGP = \angle GEP + \angle EGP = 180^\circ - \angle EPG = 180^\circ - (180^\circ - \beta) = \beta. \tag{9}$$
From (8) and (9) we have (7). By the tangent–chord theorem for $(PFCG)$, $\angle(t, PF) = \angle FGP$, so the angle between t and PH is $\angle HEP$, which by the tangent–chord theorem for $(PEHH')$ means t is tangent to $(PEHH')$ at P.

Thus t is a common tangent to $(PFCG) = (CFG)$ and $(PEHH') = (HEH')$ at P. □

Therefore
$$\boxed{\text{the circumcircles of } \triangle HEH' \text{ and } \triangle CFG \text{ are tangent.}}$$

Observation. Once Claims 1, 2 and 3 are known, P also lies on (DEF) by the Miquel theorem applied to the four lines BD, BC, CD, AH. □

Problem 4.

Let $n \geq 2$ be an integer. Initially, the number 1 is written n times on a blackboard. An operation consists of choosing two numbers a and b currently on the blackboard, not both zero, and replacing them with two new numbers

$$\frac{(a-b)^2}{a+b} \quad \text{and} \quad \frac{4ab}{a+b}.$$

Determine all integers n for which it is possible, after a finite number of operations, for the number n to appear on the blackboard.

Proposed by Malaysia.

Solution 1

The answer is precisely the powers of 2.

The sum of the numbers is invariant. Initially, the sum is n, and the target state is to have the number n on the board, which means the configuration must be $\{n, 0, \ldots, 0\}$.

For a rational number x and an odd prime p, let $v_p(x)$ denote the exponent of p in the prime factorization of x. For a multiset S of rationals, define

$$V_p(S) := \min_{\substack{s \in S \\ s \neq 0}} v_p(s).$$

Claim 1. If a state S satisfies $V_p(S) < 0$ for some odd prime p, then every state S' reachable from S also satisfies $V_p(S') < 0$.

Proof. Let $u, v \in S$ be the pair replaced by u', v', and set $k = V_p(S) < 0$. If one of u, v is zero, then the multiset is unchanged, so there is nothing to prove. Thus assume u, v are both nonzero.

We show that $\min(v_p(u'), v_p(v')) \leq k$ whenever the corresponding value is nonzero, or that the minimum is still attained elsewhere in S.

- If $v_p(u) \neq v_p(v)$, by symmetry assume $v_p(u) < v_p(v)$. Then $v_p(u+v) = v_p(u)$, hence

  $$v_p(u') = 2v_p(u-v) - v_p(u+v) = 2v_p(u) - v_p(u) = v_p(u).$$

  Thus an element with valuation $v_p(u)$ persists.

- If $v_p(u) = v_p(v) = j$, then $v_p(u+v) \geq j$, so

  $$v_p(v') = v_p(4uv) - v_p(u+v) = 2j - v_p(u+v) \leq j.$$

  In particular, if u, v were the only elements with minimal valuation $j = k$, then v' has valuation $\leq k$, and the minimum is still $\leq k$ in S'.

In all cases the minimum valuation does not increase, so the property $V_p(S) < 0$ is preserved. □

The target state $T = \{n, 0, \ldots, 0\}$ satisfies $V_p(T) = v_p(n) \geq 0$ for every odd prime p. Hence, by Claim 1, no configuration occurring in a successful sequence can satisfy $V_p < 0$ for any odd prime p; equivalently, along a successful sequence every operation must yield only numbers with non-negative v_p for every odd prime.

We now analyse the permissible operations along such a successful sequence, starting from the initial configuration, where every nonzero number is a power of 2. Suppose that, before an operation, every nonzero number is a power of 2. Let a, b be the two chosen numbers.

- **One of a, b is zero.** If $a = 0$, $b = 2^k$, the operation produces $(2^k, 0)$; the multiset is unchanged.
- $a = b = 2^k$**.** Then
$$\frac{(2^k - 2^k)^2}{2^{k+1}} = 0, \qquad \frac{4 \cdot 2^{2k}}{2^{k+1}} = 2^{k+1},$$
so $\{2^k, 2^k\}$ is replaced by $\{0, 2^{k+1}\}$. All nonzero numbers remain powers of 2.
- $a = 2^k \neq b = 2^m$ **with** $k < m$**.** Then $a + b = 2^k(1 + 2^{m-k})$, where $1 + 2^{m-k}$ is an odd integer greater than 1. Pick any odd prime divisor p of $1 + 2^{m-k}$. Then $v_p(a + b) > 0$, while $v_p(4ab) = 0$, hence the new value
$$b' = \frac{4ab}{a + b} \quad \text{satisfies} \quad v_p(b') = -v_p(a + b) < 0.$$
This is forbidden in a successful sequence by Claim 1.

Hence, along a successful sequence, the only effective operation we may perform is $\{2^k, 2^k\} \to \{0, 2^{k+1}\}$. Starting from n copies of $1 = 2^0$, we successively pair equal powers of 2, and the process terminates when all nonzero numbers are distinct powers of 2. For the final state to contain a single nonzero number, n must be a power of 2.

Conversely, if $n = 2^r$, then the number n can be obtained. Indeed, repeatedly apply the operation to two equal nonzero numbers:
$$(2^k, 2^k) \mapsto (0, 2^{k+1}).$$
Starting with 2^r copies of $1 = 2^0$, pair them to obtain 2^{r-1} copies of 2, then 2^{r-2} copies of 4, and so on. Eventually we obtain one copy of $2^r = n$ and all other entries are 0.

Therefore, the required integers are exactly the powers of 2.

Solution 2

The answer is precisely the powers of 2.

Notice that the sum of the numbers stays constant, equal to n. Suppose first that it is possible to obtain n on the board. Since the numbers are always non-negative, the final configuration must then be $(n, 0, 0, \ldots, 0)$.

We work backwards. We claim that, in every configuration encountered while rolling back the operations, every number on the board is of the form $nm/2^j$ for some integers $m, j \in \mathbb{Z}_{\geq 0}$.

This holds for the final configuration $(n, 0, \ldots, 0)$, where each entry is either 0 or $n = n \cdot 1/2^0$. Suppose it holds for the current board, and let x, y on it come from a pair $a \geq b$ via the operation. Set $s := a + b = x + y$. Then $(a - b)^2 = xs$, so
$$a = \frac{s + \sqrt{xs}}{2}, \qquad b = \frac{s - \sqrt{xs}}{2}.$$

Since $s = x + y$ is a sum of two numbers of the form $nm/2^j$, s itself is of that form. Moreover, a, b are rational because we are rolling back an actual successful sequence: all numbers in the forward sequence are rational, since the operation preserves rationality from the initial rational configuration. Hence $\sqrt{xs} \in \mathbb{Q}$.

We now show that $\sqrt{xs}$ is also of the form $nm/2^j$. Write $x = nm_1/2^{j_1}$ and $s = nm_2/2^{j_2}$; then
$$xs = \frac{n^2\, m_1 m_2}{2^{j_1+j_2}}, \qquad \text{so} \qquad \sqrt{xs} = n\sqrt{\frac{m_1 m_2}{2^{j_1+j_2}}}.$$

Since $\sqrt{xs} \in \mathbb{Q}$ and $n \in \mathbb{Z}_{>0}$, the quantity $\sqrt{m_1 m_2/2^{j_1+j_2}}$ is rational; write it as p/q in lowest terms. Then $m_1 m_2 \cdot q^2 = p^2 \cdot 2^{j_1+j_2}$, and because $\gcd(p, q) = 1$ we obtain $q^2 \mid 2^{j_1+j_2}$, so q is a power of 2, say

$q = 2^a$. Hence $\sqrt{xs} = np/2^a$, which is of the form $nm/2^j$. Combining with the form of s, both a and b are of the form $nm/2^j$.

By induction, every number that appears in the rollback sequence is of the form $nm/2^j$. In particular, the initial configuration consists of 1's, so $1 = nm/2^j$ for some $m, j \geq 0$, i.e., $nm = 2^j$. Hence $n \mid 2^j$, so n is a power of 2.

Conversely, if $n = 2^r$, then the number n can be obtained. Indeed, repeatedly apply the operation to two equal nonzero numbers:

$$(2^k, 2^k) \mapsto (0, 2^{k+1}).$$

Starting with 2^r copies of $1 = 2^0$, pair them to obtain 2^{r-1} copies of 2, then 2^{r-2} copies of 4, and so on. Eventually we obtain one copy of $2^r = n$ and all other entries are 0.

Therefore, the required integers are exactly the powers of 2.