On algebraic curves and surfaces in secondary schools

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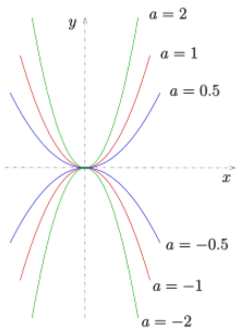
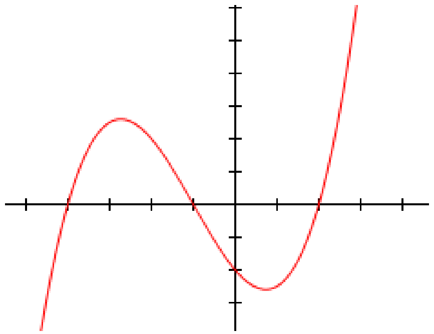
Abstract

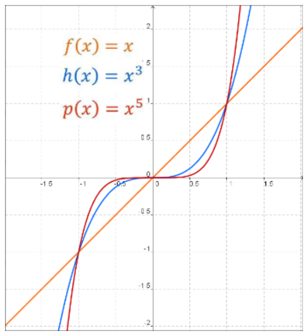
- We give some examples of real algebraic curves and surfaces which can be studied in secondary school by gifted pupils.
- This is based on personal experience with several school groups of pupils (talks and workshops) in Germany, France, Portugal and Greece.
- For the visualization of algebraic surfaces, the free software SURFER is used.
- SURFER was developped for the Year of Mathematics in Germany 2008 as part of the MFO‘s touring exhibition IMAGINARY.
- IMAGINARY has developed to a highly successful, international RPA (Raising Public Awareness) network in mathematics which is an independent non-for-profit company now. SURFER can be downloaded from   https://imaginary.org/de

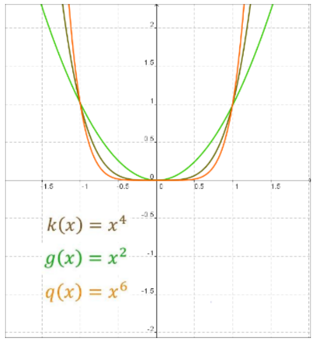
Functions of one variable and their graphs

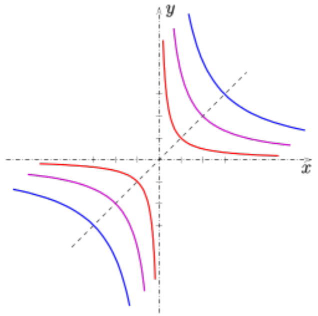
- Straight **line**: $y = ax+b$
- Standard **parabola**: $y = x^2$
- General parabola: $y = ax^2+bx+c$
- Parabola with two zeros: $y = (x-a)(x-b)$
- **Cubic Parabola**: $y = ax^3+bx^2+cx+d$
- Special cubic parabola: $y = x^2(x-a)+b$
- **Higher powers**: $y = x^{2n}$ or $y = x^{2n+1}$
- **Hyperbola**: $y = 1/x$ or $xy = 1$





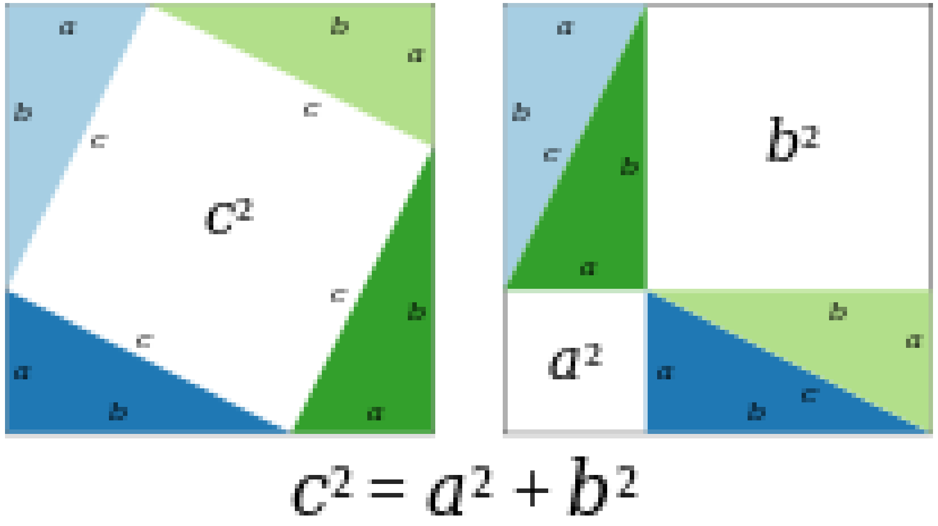


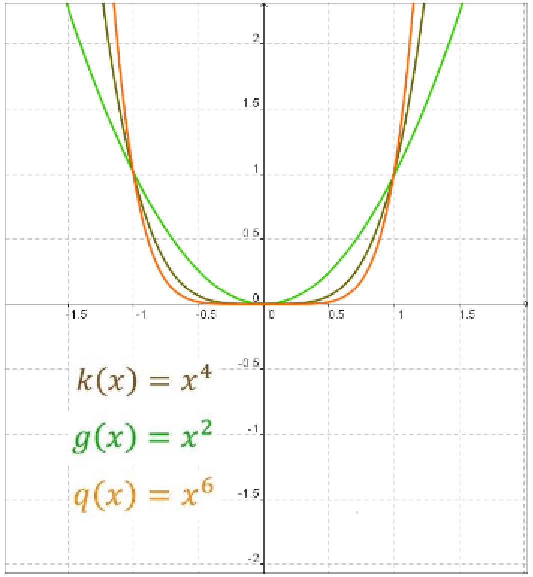




Implicit functions: the circle and the square

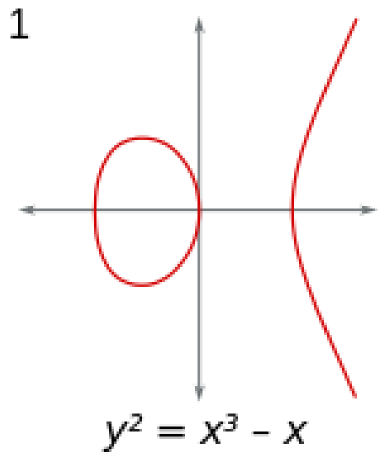
- **Theorem of Pythgoras**:
  Distance d of point (x,y) to origin (0,0) satisfies $d^2 = x^2+y^2$
- **Circle**: $x^2+y^2 = 0$
- **Square** approximation: $x^4+y^4 = 0$
- Better approximation: $x^{2n}+y^{2n} = 0$ (the higher is n)

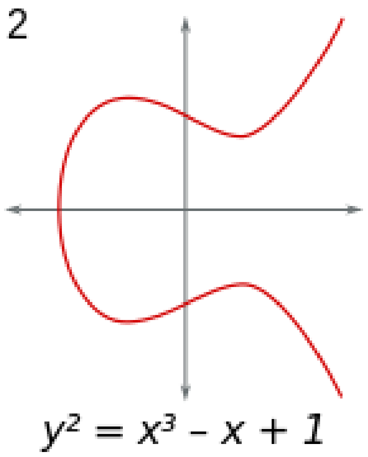


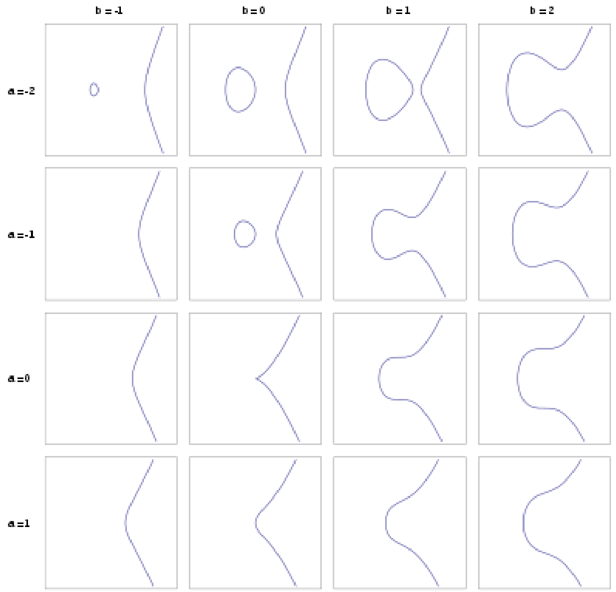


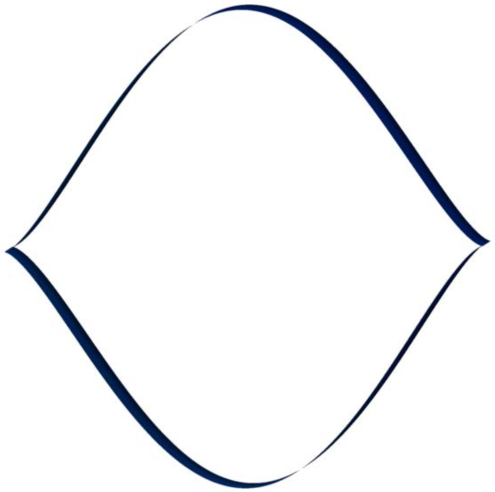
Elliptic curves and other square roots

- Square root of a function f(x): $y = \sqrt{f(x)}$, defined for $f(x) \geq 0$
- Equivalent equation: $y^2 - f(x) = 0$
- **Elliptic curve**: $y^2 - x^2(x-a) - b = 0$
- **Lemon curve**: $y^2 + (x-a)^3(x+a)^3 = 0$



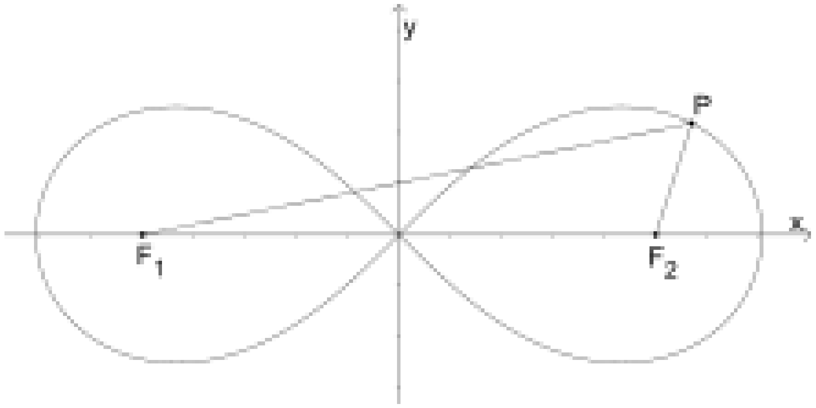


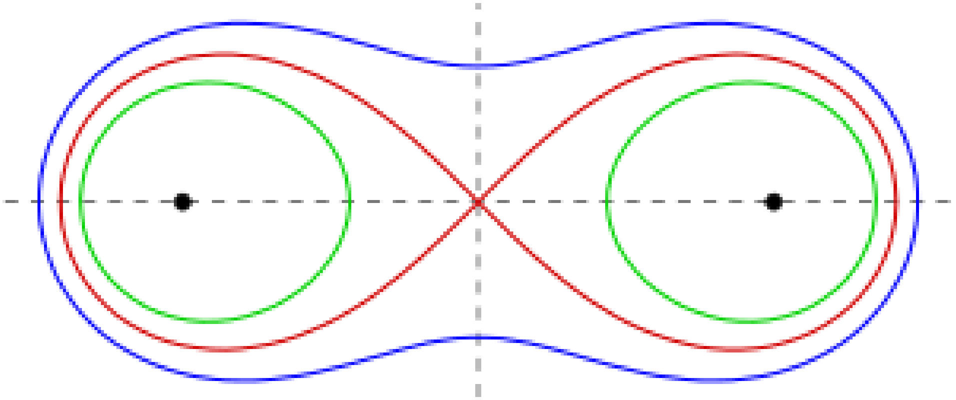




The lemniscate

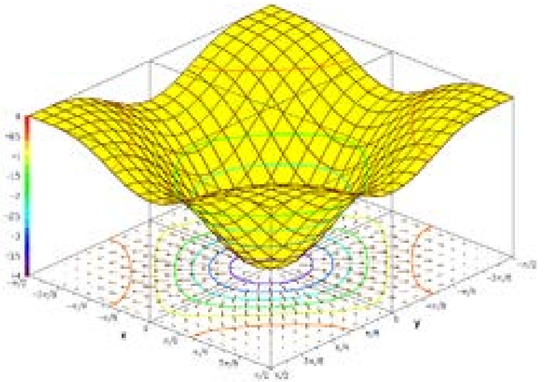
- Definition of the ellipse: set of all points, such that the **sum** of distances to two fixed points (a,0) and (-a,0) is constant.
- Thus: $\sqrt{(x-a)^2+y^2} + \sqrt{(x+a)^2+y^2} = b$
- Show that this gives equation of an ellipse: $Ax^2+By^2 = 1$ (lengthy!)
- Definition of the lemniscate: set of all points, such that the **product** of distances to two fixed points (a,0) and (-a,0) is constant.
- This gives the **lemniscate-equation**: $((x-a)^2+y^2)((x+a)^2+y^2) = b^2$

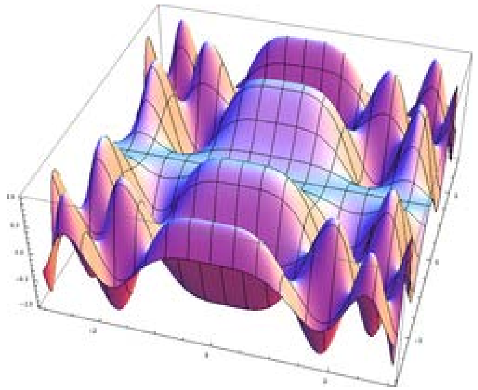




Functions of two variables and their graphs

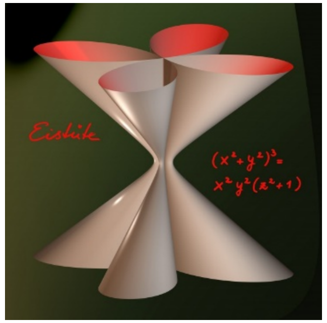
- Graph of $z = f(x,y)$ („**explicit function**“) is 2-dimensional curved surface
- Example: Plane $z = ax+by+c$
- Even more interesting: graphs of **implicit functions** $f(x,y,z) = 0$
- Consider the set of solutions of such an equation, interpret them as points with coordinates (x,y,z) in 3D-space
- **Algebraic geometry**: $f(x,y,z)$ is a polynomial $p(x,y,z)$
  study the geometric properties of the set of solutions $p(x,y,z) = 0$
- Generically, $p(x,y,z) = 0$ gives a 2-dimensional object (a surface)

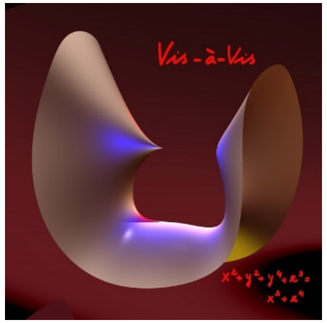


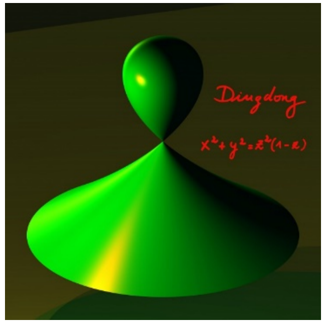


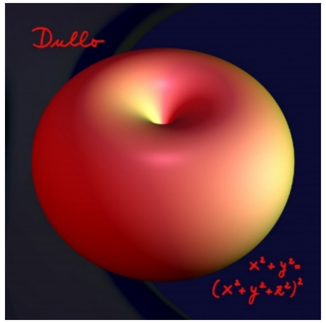
The SURFER software

- Software is adaption of research software in real algebraic geometry
- **Real-time visualization** of the zero set of a real polynomial p(x,y,z)
- Polynomial degrees up to 20, four parameters
- Can easily be used by non-mathematicians
- SURFER was developed by MFO and three universities for the Year of Mathematics in Germany 2008
- Additional features:
  rotation with mouse, scaling, coulors, layers, short movies



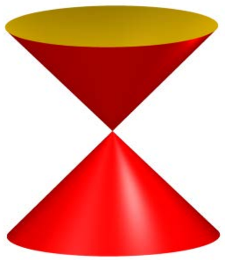


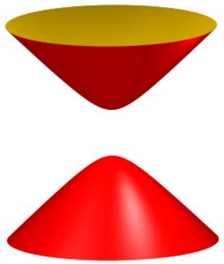




Implicit functions: cylinder, sphere and double cone

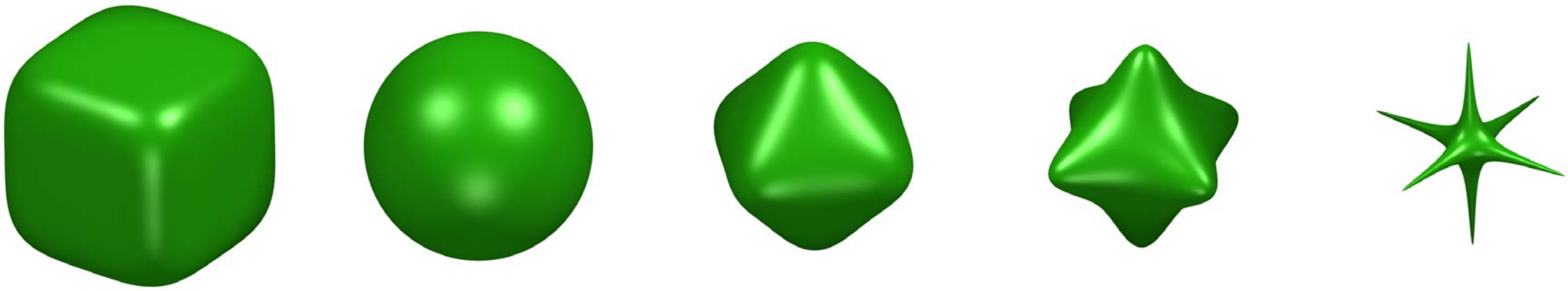
- **Cylinder** of radius a: $x^2+y^2 = a^2$
  (because x^2+y^2 is the square of the distance of (x,y,z) to the z-axis)
- **Sphere** of radius a: $x^2+y^2+z^2 = a^2$
  (because $x^2+y^2+z^2$ is the square of the distance of (x,y,z) to the origin)
- **Double cone**: $x^2+y^2-z^2 = 0$
  (the equation $x^2-z^2 = (x+z)(x-z) = 0$ gives a cross and x^2+y^2 has rotational symmetry around the z-axis)
- **Hyperboloid** of one ($a>0$) or two ($a<0$) sheets:
  $x^2+y^2-z^2 = a$





Cube and octahedron

- Cube: $x^4+y^4+z^4 = 1$
- Sphere: $(x^2+y^2+z^2)^2 = 1$
- **Sphere – cube – octahedron**: $x^4+y^4+z^4 +2a(x^2y^2+x^2z^2+y^2z^2)= 1$

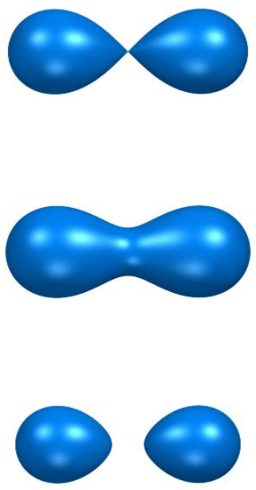


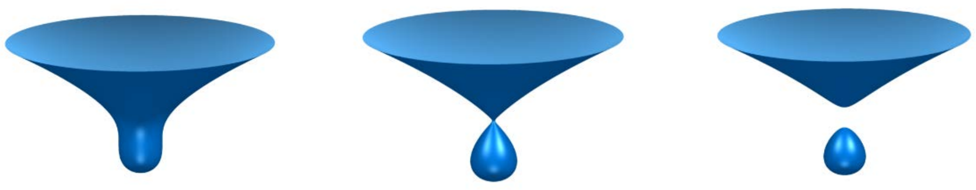
Further constructions: union, intersection and rotational symmetry

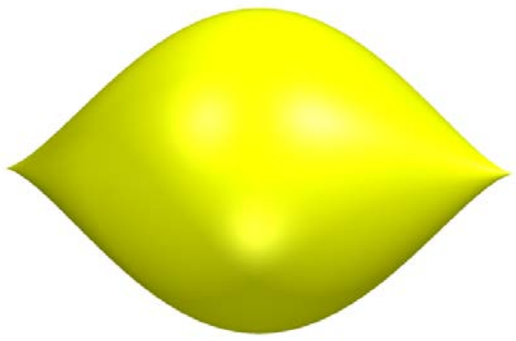
- The **union** of two surfaces is given by the product of their polynomials: $p(x,y,z)q(x,y,z) = 0$
- The **intersection** of two surfaces is (generically) a 1-dimensional object, given by $p(x,y,z)^2 + q(x,y,z)^2 = 0$. As a 1-dimensional object, SURFER is not suitable to visualize it.
- A **small tube** of size **$a>0$ around the intersection** is given by the following equation: $p(x,y,z)^2 + q(x,y,z)^2 = a$
- If an equation $p(x,y,z) = 0$ has **singularities**, these can be **smoothened** by a small deformation $p(x,y,z) = a$.
- As $x^2 + y^2$ is the square of the distance of a point (x,y,z) to the z-axis, an equation of the form $p(x^2 + y^2, z) = 0$ has **rotational symmetry** around the z-axis.

Rotational Symmetry (2)

- **Rotated lemniscate**: $((x-a)^2+y^2+z^2)((x+a)^2+y^2+z^2) = b^2$
  (looks like a chemical p-orbital)
- **Rotated elliptic curve**: $y^2 + z^2 - x^2(x-a) - b = 0$
  (looks like a water drop)
- **Rotated lemon curve**: $y^2 + z^2 + (x-a)^3(x+a)^3 = 0$
  (looks like a lemon)

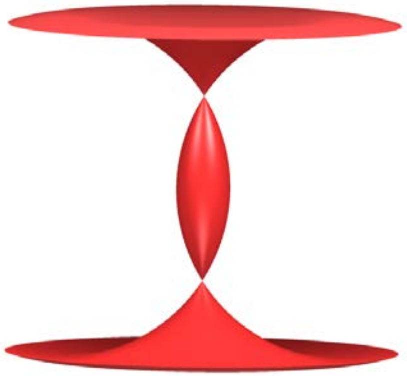






Rotational Symmetry (3): the torus

- Construction of a **torus**:
  rotate a circle $(t-a)^2+z^2 = b^2$ around the z-axis, i.e. $t^2 = x^2+y^2$
- Thus $t^2-2at+a^2+z^2 = b^2$, insert t^2, bring $2at$ to the right
- Gives $x^2+y^2+z^2+a^2-b^2 = 2at$, take square of this equation, insert t^2
- Yields **torus-equation**: $(x^2+y^2+z^2+a^2-b^2)^2 = 4a^2(x^2+y^2)$
- Big radius a, small radius b, but works also for $a \leq b$ (singularities!)

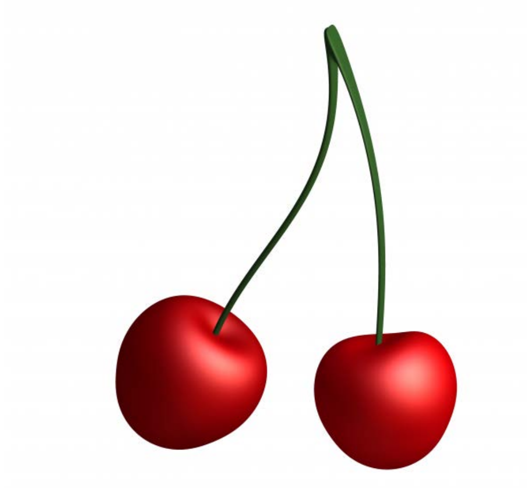


The Year of Mathematics in Germany 2008

- Several scientific years in Germany, e.g. 2005 Einstein-Year
- In **2008: Year of Mathematics**
- was the **most successful** of all scientific years in Germany up to now! (in terms of number of activities and visitors)
- Contribution of MFO: touring exhibition **IMAGINARY**, which has developped to an international RPA network for mathematics since then, now an independent non-for-profit company
- In 2008 **competition** to the general audience in Germany via ZEIT and SPEKTRUM (newspaper/serial) on interesting SURFER pictures
- More than 10.000 submissions in one month!
- Many activities since 2008 with more than 2.000.000 real and virtual international visitors in total (https://imaginary.org/de)

Example: Valentina Galata

- Valentina Galata was a 16-year old pupil (in 2008) at a secondary school
- She won several prizes for creating many astonishing SURFER formulas and pictures of „**real things**“
- Her formulas and pictures can be found in a gallery at IMAGINARY

Valentina Galata: cup of coffee

- Formula:
- 0=a∗((−1∗y+4)3−8)3−1∗a∗(x2+z2−1)8+64
- 0=(z2)+x2−10+((3∗y−7)5)
- 0=((x−3)2+6∗(x−3)+9+(y+2)2−2∗(y+2)∗(z+1.1−1∗y∗1.2)2+(z+1.1−1∗y∗1.2)4−0.89∗c+c∗(b/2)∗(x−3)−2∗b∗c+b∗(y+2.3)+(c/3)∗(z+1.1−1∗y∗1.7)+1.1∗b)
- 0=z2+x2+(y+2.7)4+5∗(y+2.7)3+6∗(y+2.7)2−4∗(y+2.7)−8
- 0=x2+y2+z2−1
- 0=((x−2.4)2+(y+3)2+(z+0.9)2−0.1)∗((x−1)2+(y+3)2+(z−2.3)2−0.08)∗((x+1)2+(y+3.8)2+(z−1)2−0.06)∗((x+1)2+(y+3.3)2+(z+1.8)2−0.2)
- 0=((x+2.7)2+(y−2)2+(z−1.3)2−0.09)∗((x−0.1)2+(y−2.5)2+(z−2.2)2−0.2

Thank you for your attention!

Note: Pictures of graphs were taken from Wikipedia
Other pictures: SURFER, Imaginary, MFO